

Continuous-Time Consensus under Non-Instantaneous Reciprocity

Samuel Martin and Julien M. Hendrickx

Abstract—We consider continuous-time consensus systems whose interactions satisfy a form of reciprocity that is not instantaneous, but happens over time. We show that these systems have certain desirable properties: They always converge independently of the specific interactions taking place and there exist simple conditions on the interactions for two agents to converge to the same value. This was until now only known for systems with instantaneous reciprocity. These results are of particular relevance when analyzing systems where interactions are a priori unknown, being for example endogenously determined or random. We apply our results to an instance of such systems.

I. INTRODUCTION

We consider systems where n agents each have a value $x_i \in \mathbb{R}$ that evolves according to

$$\dot{x}_i = \sum_{j=1}^n a_{ij}(t)(x_j(t) - x_i(t)), \quad (1)$$

where the $a_{ij}(t) \geq 0$ are non-negative functions of time. This means that the value of x_i is continuously attracted by the values of the agents j for which $a_{ij}(t) \neq 0$. These systems are called consensus systems because the interactions tend to reduce the disagreement between the interacting agents, and because any consensus state where all x_i are equal is an equilibrium of the system. Analogous systems also exist in discrete time [1]–[3]. Consensus systems play a major role in decentralized control [4], data fusion [5], [6] and distributed optimization [7], [8], but also when modeling some animal [9], [10] or social phenomena [11], [12].

General convergence results for consensus systems involve connectivity assumptions that are hard to check for state-dependent interactions, and do not allow treating clustering phenomena. As detailed in the state of the art, more recent results guarantee convergence to one or several clusters under various assumptions on the symmetry or reciprocity of the interactions. All these reciprocity properties have however to

be satisfied instantaneously and at every time. We extend them to treat systems where reciprocity is not instantaneous but happens on average over time.

This extension only holds under certain assumptions on the way reciprocity occurs. Indeed, non-instantaneous reciprocity may fail to ensure convergence and lead to oscillatory behaviors when the interaction weights are not properly bounded, or when the time periods across which it occurs grow unbounded (see Section III-B for an example). To prove our result we show that, for an appropriate sequence of times t_k , the states $x(t_k)$ can be seen as the trajectory of a certain discrete time consensus system. By analyzing the effect of each matrix of this system on some artificial initial conditions, we obtain bounds on their coefficients, and show that this system satisfies reciprocity conditions guaranteeing convergence.

The rest of the paper is organized as follows. The introduction includes a state of the art on consensus systems, a subsection pointing out the interest of non-instantaneous reciprocity and a summary of our contributions. Section II formally introduces the system that we are considering and presents our main results. Examples illustrating our results and the necessity of an underlying assumption are then presented in Section III. In Section IV, we demonstrate the use of our results on a specific multi-agent applications. Sections V and V-B contain the proofs, and we finish by some conclusions in Section VI.

State of the art

Consensus systems have been the object of many studies during the recent years, focusing particularly on finding conditions under which the system converges, possibly to a consensus state, and also on the speed of convergence. Classical results typically guarantee convergence to consensus under some (repeated) connectivity conditions on the interactions, see for example [1], [3], [13] or [14], [15] for surveys.

A variation of this repeated connectivity condition was also recently proposed in [16] for certain classes of state-dependent interactions where the attraction magnitude should be non-decreasing with distance between agents' positions. It involves a graph defined by connecting a node to another when the dynamics of the former is sufficiently and repeatedly influenced by the latter and this being true for all positions of the two agents.

Different recent works have shown that stronger results hold when the interactions satisfy some form of reciprocity. Hendrickx and Tsitsiklis have for example introduced the *cut-balance* assumption on the interactions [17], stating that there

Julien Hendrickx is with the ICTEAM institute, Université catholique de Louvain, Louvain-la-Neuve, Belgium. julien.hendrickx@uclouvain.be

Samuel Martin is with Université de Lorraine and CNRS, CRAN, UMR 7039, 2 Avenue de la Forêt de Haye, 54518 Vandœuvre-lès-Nancy, France (part of the work was carried out when S. M. was with the ICTEAM institute). samuel.martin@univ-lorraine.fr

This work is supported by the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Program, initiated by the Belgian Science Policy Office, and by the Concerted Research Action (ARC) of the French Community of Belgium. This work is also partly supported by the French Agence Nationale de la Recherche under ANR COMPACS - Computation Aware Control Systems, ANR-13-BS03-004 and by the CNRS via the interdisciplinary PEPS Project MADRES.

exists a K such that for every subset S of agents and time t , there holds

$$\sum_{i \in S, j \notin S} a_{ij}(t) \leq K \sum_{i \in S, j \notin S} a_{ji}(t). \quad (2)$$

This assumption can actually be shown to mean that whenever an agent i influences agent j indirectly, agent j also influences agent i indirectly, with an intensity that is within a constant ratio of that of i on j . Particular cases of this assumptions include symmetric interactions $a_{ij} = a_{ji}$, bounded-ratio symmetry $a_{ij} \leq K a_{ji}$, or any average-preserving dynamics $\sum_j a_{ij} = \sum_j a_{ji}$ for every i . It was shown in [17] that systems satisfying the cut-balance assumption (2) always converge, though not necessarily to consensus. Moreover, two agents' values converge to the same limiting value if they are connected by a path in the graph of *persistent* interactions (also called *unbounded* interactions in the literature), defined by connecting i and j if $\int_0^\infty a_{ij}(t)dt$ is infinite. These results allow analyzing the convergence properties of systems with relatively complex interactions; see the discussion in [17] for an example in opinion dynamics, or [18] for an application to system involving event-based ternary control of second order agents.

Martin and Girard have later shown [19] that in the case of convergence to a global consensus, the cut-balance assumption could be weakened, allowing for the interaction ratio bound K to slowly grow with the amount of interactions that have already taken place in the system. They also provide an estimate of the convergence speed in terms of the interactions having taken place.

Related convergence results were also proved for systems involving a continuum of agents under a strict symmetry assumption in [20]. An alternative reciprocity condition called *arc-balance* was considered in [21]; it requires *all weights* $a_{ij}(t)$ to be within a constant ratio of each other, except those for which $\int_0^\infty a_{ij}(t)dt < \infty$.

Finally, we note that similar results of convergence under some reciprocity conditions have been obtained for discrete time consensus systems, see for example [3], [22]–[25]. However, none of these results allow for non-instantaneous reciprocity.

Non-instantaneous reciprocity

All the results taking advantage of reciprocity require the reciprocity condition to be satisfied instantaneously at (almost) all times. They would thus not apply to systems that are essentially reciprocal, but where the reciprocity may be delayed, or where it happens over time: In systems relying on certain wired or wireless network protocols, agents may be unable to simultaneously send and receive information, resulting in loss of instantaneous reciprocity, even if the interactions are meant to be reciprocal. Non-instantaneous reciprocity also arises in a priori symmetric systems where the control of the agents is event-triggered or self-triggered. Indeed, suppose that at some time the conditions are such that agents i and j should interact. It is very likely that one agent will update its control action

before the other, so that during a certain interval of time the actual interactions will not be symmetric.

Similar problems are present in systems prone to occasional failures, or unreliable communications, where the communication between two agents can temporarily be interrupted in one direction for a limited amount of time.

Issues with non-instantaneous reciprocity may also arise in swarming processes or any multi-agent control problem where sensors have a limited scope. Suppose indeed that the sensors are not omnidirectional, as it is for example the case for human or animal eyes. It is then generally impossible for an agent to observe all its neighbors at the same time. The same issue arises if the agent can only treat a limited number of neighbors simultaneously. A natural solution is then to observe a subset of the neighbors and to periodically modify the subset being observed. This can for example be achieved by continuously rotating the directions in which observations are made. In that case, even if the neighborhood relation is symmetrical, it is again highly likely that an agent i will sometime observe an agent j without that j is observing i at that particular moment, but that j will observe i later. In all these situations, one could hope to take advantages of the essential reciprocity of the system design even if this reciprocity is not always instantaneously satisfied.

Contributions

We show in our main result (Theorem 1) that the convergence of systems of the form (1) is still guaranteed if the system satisfies some form of non-instantaneous reciprocity, or reciprocity on average. More specifically, we assume that the cut-balance condition (2) is satisfied *on average* on a sequence of contiguous intervals. These intervals can have arbitrary lengths, but the amount of interaction taking place during each of them should be uniformly bounded. Under these assumptions, we show that the system always converges. Moreover, two agent values converge to the same limit if they are connected by a path in the graph of persistent interactions, defined by connecting two agents i, j if $\int_{t=0}^\infty a_{ij}(t)dt$ is infinite.

We also particularize our general result to systems satisfying a form of pairwise reciprocity over bounded time intervals. This particularized result is more conservative, but its condition can often be easier to check. We illustrate it on an application.

II. PROBLEM STATEMENT AND MAIN RESULTS

We study the integral version of the consensus system (1):

$$x_i(t) = x_i(0) + \int_0^t \sum_{j=1}^n a_{ij}(s)(x_j(s) - x_i(s))ds, \quad (3)$$

where for all $i, j \in \mathcal{N} = \{1, \dots, n\}$, the *interaction weight* a_{ij} is a non-negative measurable function of time, summable on bounded intervals of \mathbb{R}^+ . There exists a unique function of time $x : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ which satisfies for all $t \in \mathbb{R}^+$ the integral equation (3), and it is locally absolutely continuous

(see Theorem 54 and Proposition C.3.8 in [26, pages 473-482]). This function is actually the Caratheodory solution to the differential equation (1) and can equivalently be defined as absolutely continuous function satisfying (1) at almost all times. We call it the *trajectory* of the system.

Following the discussion in the Introduction, we introduce a new condition generalizing Condition 2 by allowing for non-instantaneous reciprocity of interactions; we only require that the reciprocity occurs on the integral weights $\int a_{ij}(s)ds$ over some bounded time intervals.

Assumption 1 (Integral weight reciprocity): There exists a sequence $(t_p)_{p \in \mathbb{N}}$ of increasing times with $\lim_{p \rightarrow +\infty} t_p = +\infty$ and some uniform bound $K \geq 1$ such that, for all non-empty proper subsets S of \mathcal{N} , and for all $p \in \mathbb{N}$, there holds

$$\sum_{i \in S, j \notin S} \int_{t_p}^{t_{p+1}} a_{ij}(t)dt \leq K \sum_{i \in S, j \notin S} \int_{t_p}^{t_{p+1}} a_{ji}(t)dt. \quad (4)$$

We will see in a simple example in Section III-B that Assumption 1 alone is not sufficient to guarantee the convergence of the system. We need to further assume that the integral of the interactions taking place in each interval $[t_p, t_{p+1}]$ is uniformly bounded.

Assumption 2 (Uniform upper bound on integral weights): The sequence (t_p) used in Assumption 1 is such that

$$\int_{t_p}^{t_{p+1}} a_{ij}(t)dt \leq M,$$

holds for all $i, j \in \mathcal{N}$, $p \in \mathbb{N}$ and some constant M .

We now state our main result, whose proof is presented in Section V-A.

Theorem 1: Suppose that the interaction weights of system (3) satisfy Assumptions 1 (integral reciprocity) and 2 (upper bound on weight integral). Then, every trajectory x of system (3) converges.

Moreover, let $G = (\mathcal{N}, E)$ be the graph of persistent weights defined by connecting (j, i) if $\int_0^\infty a_{ij}(t)dt = +\infty$. Then, there is a directed path from i to j in G if and only if there is a directed path from j to i , and there holds in that case $\lim_{t \rightarrow \infty} x_i(t) = \lim_{t \rightarrow \infty} x_j(t)$.

The second part of the theorem implies that there is a local consensus in each strongly connected component¹ of the graph G of persistent interactions. Notice that the second part of the theorem also implies that each strongly connected component is fully disconnected from the others in graph G : no edge leaves one component to arrive at another. This is due to the reciprocity Assumption 1.

Assumption 1 generalizes most (instantaneous) reciprocity conditions available in the literature, including cut-balance,

¹Strongly connected components are defined as the classes of equivalence on the node set where node i and j belong to the same class if and only if i and j are connected to each other by at least a path from i to j and a path from j to i .

and is thus automatically satisfied by any system satisfying such conditions. It is moreover satisfied by classes of systems subject to some form of reciprocity that is delayed due for example to communication constraints. It applies for instance to systems where agents engage in interaction with a neighbor while the latter may be asleep or already busy interacting with another agent. We provide an example of such application in Section IV.

There are several options to check whether a given (non-instantaneously reciprocal) system verifies the integral condition in Assumption 1. One option is to show that it is implied by the specific reciprocal nature of the system, as done in Section IV. Another one is to derive sufficient conditions on the initial configuration which implies that the integral condition remains valid over time (see for instance [27]).

The reciprocity conditions and the time intervals over which it has to be satisfied are global. We now introduce a new local assumption that we will show to imply Assumptions 1 and 2 when interactions are bounded. It requires that whenever an agent j influences an agent i at some time t , both agents should influence each other with a sufficient strength across a certain time interval around t .

Assumption 3 (Pairwise reciprocity): There exists a constant $\varepsilon > 0$ such that for every unordered pair $\{i, j\}$ with $i, j \in \mathcal{N}$ distinct, there exists a constant $T_{ij} > 0$ such that for all $t \geq 0$, if $a_{ij}(t) > 0$ or $a_{ji}(t) > 0$, then there exists $\underline{t}_{ij}, \bar{t}_{ij}$ such that

- a) $\bar{t}_{ij} - \underline{t}_{ij} \leq T_{ij}$,
- b) $t \in [\underline{t}_{ij}, \bar{t}_{ij}]$,
- c) $\int_{\underline{t}_{ij}}^{\bar{t}_{ij}} a_{ij}(t)dt \geq \varepsilon$ and $\int_{\underline{t}_{ij}}^{\bar{t}_{ij}} a_{ji}(t)dt \geq \varepsilon$.

Assumption 3 provides a way of verifying non-instantaneous reciprocity entirely locally, by considering separately each pair of nodes. For instance, reciprocal weights of type $a_{ij}(t) = 1 + (-1)^{\lfloor t \rfloor}$ and $a_{ji}(t) = 1 + (-1)^{\lfloor \omega t + \gamma \rfloor}$ satisfy the pairwise non-instantaneous reciprocity for any constants $\omega > 0, \gamma \in \mathbb{R}$, although one of the weights may be null while the other is not. The following Theorem is proved in Section V-B.

Theorem 2: Suppose that the interaction weights $a_{ij}(t)$ of system (3) satisfy Assumption 3 and are uniformly bounded above by some constant M' . Then they satisfy Assumptions 1 and 2, and the conclusions of Theorem 1 hold.

Remark 1: Theorem 1 and Theorem 2 are stated for systems where the coefficients $a_{ij}(t)$ only depend on time, and the proof of Theorem 1 actually uses that fact. However, these results can directly be extended to solutions of systems with state-dependent coefficients $\tilde{a}_{ij}(t, x)$, with typically $a_{ij}(t, x)$ depending on x_i and x_j . Indeed, suppose that x is a solution of

$$\dot{x}_i(t) = x_i(0) + \int_0^t \tilde{a}_{ij}(s, x(s))(x_j(s) - x_i(s))ds, \quad (5)$$

then x is also a solution of the linear time-varying systems (3) with ad hoc coefficients $a_{ij}(t) = \tilde{a}_{ij}(t, x(t))$, and Theorem

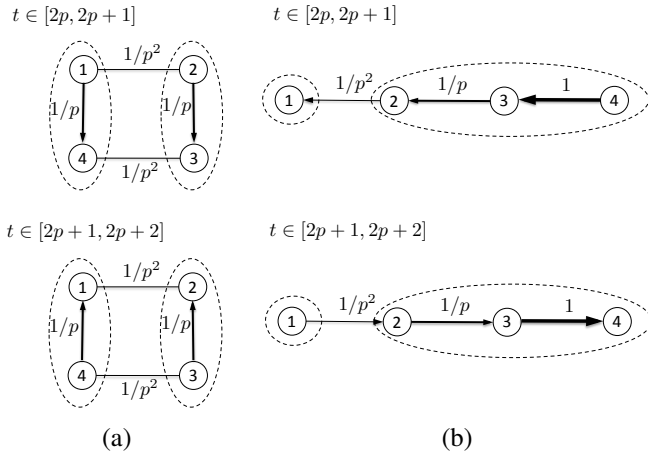


Fig. 1. Representations of the interactions taking place in example 1 (a) and in example 2 (b) in Section III-A, and of the connected components of the graph of persistent interactions, in which local consensus occur.

1 applies to that linear time-varying system. Verifying if such nonlinear systems satisfy Assumption 1 can be achieved when the structure of the interactions guarantees a sufficient reciprocity. We will see on an example in Section IV how this can be done. Note also that the existence or uniqueness of a solution to nonlinear systems of the form (5) is in general a complex issue. Similar extensions apply to randomized weights a_{ij} .

Finally, one can verify that Theorem 1 and Theorem 2 can be extended to systems with agent values x_i in \mathbb{R}^n provided that the weights a_{ij} remain scalar. It suffices indeed in that case to apply the result separately to each component of the states x_i .

III. EXAMPLES

A. System with non-instantaneous reciprocity

In this subsection, we present two simple 4-agent systems whose convergence can be established by Theorem 1 and by no other result on consensus available in the literature.

Example 1:

Our first example is depicted in Fig. 1(a). It contains two weakly interacting subsystems, inside each of which two agents successively attract each other. More specifically, the interactions start at time $t = 2$ and are defined as follows: For every $p \geq 1$,

- if $t \in [2p, 2p+2]$, $a_{12} = a_{21} = a_{34} = a_{43} = 1/p^2$,
- if $t \in [2p, 2p+1]$, $a_{32} = a_{41} = 1/p$,
- if $t \in [2p+1, 2p+2]$, $a_{23} = a_{14} = 1/p$,

and all values of $a_{ij}(t)$ that are not explicitly defined are equal to 0. One can verify that this system satisfies Assumptions 1 and 2 with $t_p = 2p$, $K = 1$ and $M = 2$. We can thus apply Theorem 1 to establish its convergence. The graph of persistent interactions can also easily be built and contains the edges $(2, 3), (3, 2), (1, 4)$ and $(4, 1)$. There are thus two connected components $\{2, 3\}$ and $\{1, 4\}$, and two local consensus $x_2^* = x_3^*$ and $x_1^* = x_4^*$.

On the other hand, notice that the system does not satisfy any instantaneous reciprocity condition, so none of available reciprocity-based results applies. Moreau's result does not apply either due to the weak interactions in $1/p^2$ between the subsystems (the interactions are not lower bounded; see Section 3.3 in [19] for a detailed explanation), and because it can only imply convergence to a global consensus while this system produces two local consensus. Observe also that our result also applies if the interactions are interrupted during arbitrarily long periods. Suppose indeed that the interactions defined above do not take place during the intervals $[2p, 2p+1]$ and $[2p+1, 2p+2]$ but during the intervals $[p^2, p^2+1]$ and $[p^2+p, p^2+p+1]$. Assumptions 1 and 2 still apply with $t_p = p^2$.

Example 2:

The second example involves a chain of four agents, which are attracted by their higher index neighbor for $t \in [2p, 2p+1]$ and their lower index neighbor for $t \in [2p+1, 2p+2]$, as depicted in Fig. 1(b). Moreover, the ratios between weights of the different interactions grow unbounded.

Specifically, the interactions start again at $t = 2$, and for each $p \geq 1$,

- if $t \in [2p, 2p+1]$, $a_{12} = 1/p^2$, $a_{23} = 1/p$ and $a_{34} = 1$
- if $t \in [2p+1, 2p+2]$, $a_{21} = 1/p^2$, $a_{32} = 1/p$ and $a_{43} = 1$

and all values of $a_{ij}(t)$ that are not explicitly defined are equal to 0. One can verify again that Assumptions 1 and 2 hold with $t_p = 2p$, $K = 1$ and $M = 2$, so that the convergence of the system follows from Theorem 1. The graph of persistent interactions contains the edges $(2, 3), (3, 2), (3, 4)$ and $(4, 3)$, resulting in a local (trivial) consensus of agent 1, and a consensus between agent 2, 3 and 4.

Again, the system satisfies no instantaneous reciprocity condition, so none of available reciprocity-based results applies. Moreover, all the results of which we are aware and that do not rely on reciprocity require the interaction to be bounded from above and from below, and establish convergence to a global consensus (see [28] for example). Since the ratios between the values of a_{34}, a_{43} and a_{32}, a_{23} grow unbounded and the system produces again two local consensus, it would thus be impossible to apply them. This remains the case even if we restrict our attention to the connected component $\{2, 3, 4\}$ and/or re-scale the values of the coefficients by scaling time.

Besides, Theorem 1 would again apply exactly in the same way if the interactions were interrupted during arbitrary long periods of time

B. Oscillatory behavior under integral reciprocity - Necessity of Assumption 2.

The following Proposition formalizes the fact that Assumption 1 alone is not sufficient to guarantee convergence.

Proposition 3: There exist systems of the form (3) satisfying Assumption 1 (integral reciprocity) and that admit non-converging trajectories.

To prove the Proposition, we present a 3-agent system which satisfies Assumption 1 (reciprocity) but whose trajectory does not converge. The idea is to have agent 2 oscillating between agents 1 and 3 that successively attract the former while remaining at a certain distance from each other, as depicted in Fig. 2. Agent 1 starts influencing 2. Since we only impose integral reciprocity, a_{12} and a_{21} do not have to be non-zero simultaneously. Also, because there is no uniform bound on influence, the distance between 2 and 1 has become arbitrarily close to 0 when agent 2 starts influencing back. So the overall influence of agent 2 over 1, this is $\int a_{12} \cdot (x_2 - x_1) dt$ over some time interval, can also be made arbitrarily small. This leads to an actual influence of 1 over 2 but not of 2 over 1. The same happens between 3 and 1, leading to convergence of 1 and 3 to distinct limits and oscillations of 2. We now present the formal proof.

Proof: Let $(\rho_p)_{p \in \mathbb{N}}$ be a non-decreasing sequence such that $\rho_p \geq 1$, for all $p \in \mathbb{N}$. Let us consider a system with 3 agents where $x_1(0) = 0$, $x_2(0) = 1/2$ and $x_3(0) = 1$ and with the dynamics given by system (3) with weights

$$\begin{cases} \text{if } t \in [4p, 4p+1), & a_{21}(t) = \rho_p, \\ \text{if } t \in [4p+1, 4p+2), & a_{12}(t) = \rho_p, \\ \text{if } t \in [4p+2, 4p+3), & a_{23}(t) = \rho_p, \\ \text{if } t \in [4p+3, 4p+4), & a_{32}(t) = \rho_p, \end{cases}$$

where only the non-zero weights have been detailed. Fig. 2 illustrates the dynamics of this system.

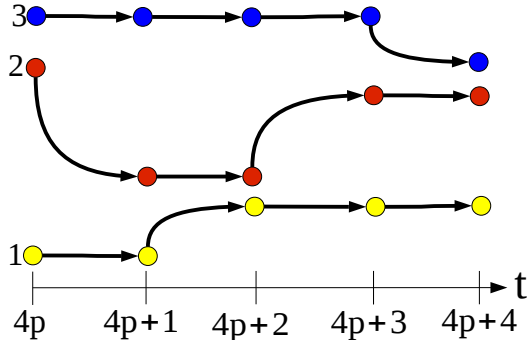


Fig. 2. Dynamics of the 3-agent system.

Here, Assumption 1 holds with $K = 1$ for $t_p = 4p$. It is easy to see that $x_1(t)$ is non-decreasing, $x_3(t)$ is non-increasing and $x_1(t) \leq x_2(t) \leq x_3(t)$ for all $t \geq 0$. Integrating the dynamics of the system, we can show that for all $p \in \mathbb{N}$:

$$\begin{aligned} x_1(4p+4) &= x_1(4p+2) \leq x_2(4p+2) = x_2(4p+1) \\ &= (1 - e^{-\rho_p})x_1(4p) + e^{-\rho_p}x_2(4p) \\ &\leq (1 - e^{-\rho_p})x_1(4p) + e^{-\rho_p}x_3(0), \end{aligned}$$

and that

$$\begin{aligned} x_3(4p+4) &\geq x_2(4p+4) = x_2(4p+3) \\ &= e^{-\rho_p}x_2(4p+2) + (1 - e^{-\rho_p})x_3(4p+2) \\ &\geq e^{-\rho_p}x_1(4p) + (1 - e^{-\rho_p})x_3(4p) \\ &\geq e^{-\rho_p}x_1(0) + (1 - e^{-\rho_p})x_3(4p). \end{aligned}$$

Combining the two previous results and the initial conditions gives us then

$$1 + (x_3(4p+4) - x_1(4p+4)) \geq (1 - e^{-\rho_p})(1 + (x_3(4p) - x_1(4p))).$$

We observe that the term $1 + (x_3(4p) - x_1(4p))$ remains larger than the product $(1 + (x_3(0) - x_1(0))) \prod_{p'=0}^p (1 - e^{-\rho_{p'}})$. Taking a sequence ρ_p growing sufficiently fast (and thus breaking the uniform bound Assumption 2), one can make this term converge to a value arbitrarily close to its initial value 2. Then, $(x_3(4p))$ and $(x_1(4p))$ do not converge to the same value. As a consequence, one can verify that x_2 will keep oscillating between x_1 and x_3 . Hence, the system does not converge. ■

IV. APPLICATION TO MOBILE ROBOTS WITH INTERMITTENT ULTRASONIC COMMUNICATION

In this section we apply our results to a realistic system of mobile robots evolving in the plane \mathbb{R}^2 and communicating using ultrasonic sensors. These sensors make for an affordable and thus widespread contactless mean of measuring distances [29], but are subject to certain limitation as detailed below. The objective of the group of robots is to achieve practical rendezvous, *i.e.* all robots should eventually lie in a ball of a certain maximal radius (see e.g. [30]). The robots have several functional constraints. The ultrasonic sensors in use are not accurate when measuring distances smaller than a radius $d_0 > 0$, thus we assume that the robots cannot make use of such measurements and are blind at short range. Also, the robots' engines are limited and the velocity of each robot cannot exceed a maximum of $\mu > 0$ in norm. Most importantly, in order to save energy, the robots activate their sensors intermittently, and in an asynchronous way: Robot i wakes up at every time t_k^i , and monitors its environment over the time-interval $[t_k^i, t_k^i + \delta_{\min}]$, for some $\delta_{\min} > 0$. (For simplicity, we take the same δ_{\min} for every robot, but this is not crucial for our result). In addition, we assume that the sequence (t_k^i) satisfies $t_{k+1}^i - t_k^i \in [\delta_{\min}, \delta_{\max}]$ for every $k \in \mathbb{N}$, for some $\delta_{\max} > \delta_{\min}$, and $t_0^i \leq \delta_{\max}$.

We will provide a simple control law for the robots ensuring some form of non-instantaneous reciprocity. Our result in Section II will then allow us to establish (i) the convergence of all robot positions, and (ii) asymptotic practical consensus, that is, all robots eventually lie at a distance from each other smaller than a certain threshold. This threshold is proportional to d_0 , the distance below which robots cannot sense each other. Since it converges, the system will not suffer from infinite oscillatory behaviors as in the example presented in Section III-B. To the best of our knowledge, such results cannot be obtained with any other convergence result available in the literature. One reason for this is that most results on consensus in the literature apply to systems which converge to a single consensus. This is clearly not the case for the system considered here since agents stop interacting at short distance.

Our control law can be expressed as the following saturated consensus equation:

$$\dot{x}_i(t) = \text{sat} \sum_{j \in \mathcal{N}} b_{ij}(t)(x_j(t) - x_i(t)), \quad (6)$$

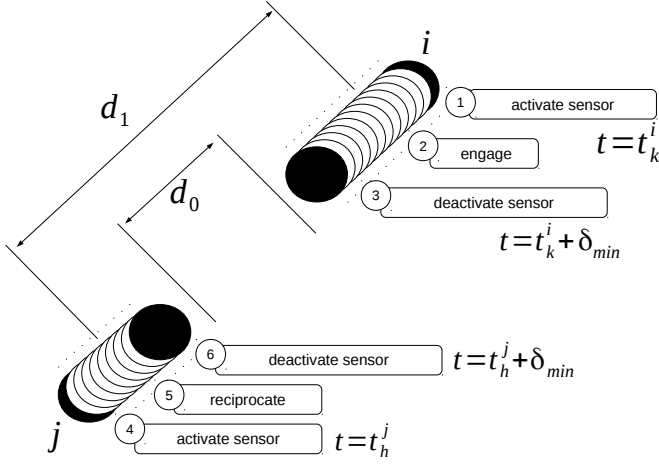


Fig. 3. Representations of the interactions taking place in the group of mobile robots with intermittent ultrasonic communication presented in Section IV. Events 1, 2 and 3 occur successively and so do events 4, 5 and 6. Event 4 occurs after event 1 and the following condition holds : $t_h^j \in [t_k^i, t_k^i + \delta_{\max}]$. When 2 occurs, $a_{ij}(t) > 0$ and when 4 occurs $a_{ji}(t) > 0$. Proposition 4 provides conditions which guarantee that event 4 always takes place when event 2 has occurred, this ensures interaction reciprocity.

where the $b_{ij}(t)$ will be specified later, and the function $\text{sat} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined by

$$\text{sat}(x) = \begin{cases} \mu \cdot \frac{x}{\|x\|} & \text{if } \|x\| \geq \mu \\ x & \text{otherwise.} \end{cases}$$

The saturation guarantees that the magnitude of the velocity of each robot remains below its limit. We now explicit how the interaction weights b_{ij} are set. The idea is represented in Fig. 3: For $t \in [t_k^i, t_k^i + \delta_{\min}]$, agent i monitors its environment. At this time, agent i sets $b_{ij}(t)$ to 1 whenever either one of the two following situations occurs : 1) its distance to j is larger than some appropriate radius $d_1 > d_0$ (*engage*), or 2) its distance to j is larger than d_0 and j has recently been influenced by i ($b_{ji} = 1$) because j was at a distance larger than d_1 from i at that time (*reciprocate*). The latter part of the algorithm is designed to ensure reciprocity, and the presence of d_1 is needed to ensure that i and j remain sufficiently distant for measurement to be made when i or j need to reciprocate.

Formally, we set $b_{ij}(t) = 0$ by default, and set it to 1 in two cases:

i engages

$$\exists k \in \mathbb{N}, (t \in [t_k^i, t_k^i + \delta_{\min}] \text{ and } \|x_i(t_k^i) - x_j(t_k^i)\| \geq d_1), \quad (7)$$

i reciprocates

$$\begin{aligned} & \exists h \in \mathbb{N}, \\ & \begin{cases} t \in [t_h^j, t_h^j + \delta_{\min}] \text{ and } \|x_i(t) - x_j(t)\| \geq d_0 \text{ and} \\ \exists k \in \mathbb{N}, \\ t_k^i \in [t_h^j - \delta_{\max}, t_h^j] \text{ and } \|x_i(t_k^i) - x_j(t_k^i)\| \geq d_1. \end{cases} \end{aligned} \quad (8)$$

Remark 2: Condition (7) can be easily implemented. To implement Condition (8), i has to keep in memory the last

activation time t_h^j at which the distance between i and j was higher than d_1 . This could for example be achieved by having j sending a message to i at t_h^j .

Under these communication rules, we have the desired result :

Proposition 4: Consider system (6) where interaction occurs according to Conditions (7) and (8). Also assume there holds

$$4\delta_{\max} \cdot \mu \leq d_1 - d_0. \quad (9)$$

Then, the group of robots asymptotically achieves practical rendezvous: $x_i^* = \lim_{t \rightarrow \infty} x_i(t)$ exists for every $i \in \mathcal{N}$, and

$$\lim_{t \rightarrow \infty} \Delta(t) \leq d_1,$$

where $\Delta(t) = \max_{i,j \in \mathcal{N}} \|x_i(t) - x_j(t)\|$.

Proof: Observe first that system (6) can be rewritten under the form of system (3) with

$$a_{ij}(t) = \frac{\mu \cdot b_{ij}(t)}{\left\| \sum_{k \in \mathcal{N}} b_{ik}(t)(x_k(t) - x_i(t)) \right\|} \quad (10)$$

if $\left\| \sum_{k \in \mathcal{N}} b_{ik}(t)(x_k(t) - x_i(t)) \right\| \geq \mu$ and $a_{ij}(t) = b_{ij}(t)$ otherwise. Since $b_{ik}(t) = 0$ whenever $\|x_k(t) - x_i(t)\| < d_0$, a_{ij} is upper bounded and thus is a non-negative measurable function, summable on bounded intervals of \mathbb{R}^+ .

Moreover, since $\Delta(t) = \max_{i,j \in \mathcal{N}} \|x_i(t) - x_j(t)\|$ is clearly nonincreasing, it follows from the definition of $a_{ij}(t)$ that

$$a_{ij}(t) \geq b_{ij}(t) \min \left(\frac{\mu}{n\Delta(0)}, 1 \right), \quad (11)$$

where $\Delta(0)$ is the initial group diameter.

In order to apply Theorem 2, we now show that the system under intermittent ultrasonic communication described above satisfies Assumption 3 with

$$\varepsilon = \min \left(\frac{\delta_{\min}\mu}{n\Delta(0)}, \delta_{\min} \right) \text{ and } T = 2\delta_{\max}.$$

Let $t \geq 0$ such that $a_{ij}(t) > 0$. Then, $b_{ij}(t) > 0$ and at least one among Conditions (7) and (8) is satisfied. Suppose first that Condition (7) is satisfied and denote by k the integer such that $t \in [t_k^i, t_k^i + \delta_{\min}]$. Clearly, Condition (7) also holds for every $s \in [t_k^i, t_k^i + \delta_{\min}]$.

We set $\underline{t}_{ij} = t_k^i$ and $\bar{t}_{ij} = t_k^i + 2\delta_{\max} \geq t_k^i + \delta_{\min}$. Clearly, there holds $t \in [\underline{t}_{ij}, \bar{t}_{ij}]$, and $\bar{t}_{ij} - \underline{t}_{ij} \leq 2\delta_{\max} = T$, so that Conditions (a) and (b) of Assumption 3 hold. Moreover, the non-negativity of a_{ij} implies that

$$\begin{aligned} \int_{\underline{t}_{ij}}^{\bar{t}_{ij}} a_{ij}(s) ds & \geq \int_{t_k^i}^{t_k^i + \delta_{\min}} a_{ij}(s) ds \\ & \geq \min \left(\frac{\mu}{n\Delta(0)}, 1 \right) \int_{t_k^i}^{t_k^i + \delta_{\min}} b_{ij}(s) ds \\ & = \min \left(\frac{\delta_{\min}\mu}{n\Delta(0)}, \delta_{\min} \right) = \varepsilon, \end{aligned}$$

where we have used (11) and the fact that $b_{ij}(s) = 1$ for all $s \in [t_k^i, t_k^i + \delta_{\min}]$ since we have seen that Condition (7) holds for those values. There remains to prove that $\int_{t_{ij}}^{\bar{t}_{ij}} a_{ji}(s)ds \geq \varepsilon$.

Since $t_{h+1}^j - t_h^j \leq \delta_{\max}$ for all $h \in \mathbb{N}$ and $t_0^j \leq \delta_{\max}$, there exists $h \in \mathbb{N}$ such that $t_h^j \in [t_k^i, t_k^i + \delta_{\max}]$, and thus $[t_h^j, t_h^j + \delta_{\max}] \subseteq [t_k^i, t_k^i + 2\delta_{\max}] = [t_{ij}^j, \bar{t}_{ij}^j]$. We show that the *reciprocate* Condition (8) is satisfied for every $s \in [t_h^j, t_h^j + \delta_{\max}]$. The second part of the condition directly follows from $t_h^j \in [t_k^i, t_k^i + \delta_{\max}]$. For the first one, observe that $\|\dot{x}_i\| \leq \mu$ (and the same holds for j), and that $\|x_i(t_k) - x_j(t_k)\| \geq d_1$ by assumption. Therefore, for any time $s \in [t_h^j, t_h^j + \delta_{\max}] \subseteq [t_k^i, t_k^i + 2\delta_{\max}]$, we have

$$\begin{aligned} \|x_i(s) - x_j(s)\| &\geq \|x_i(t_k) - x_j(t_k)\| - 4\mu\delta_{\max} \\ &\geq d_1 - (d_1 - d_0) = d_0 \end{aligned}$$

for every $s \in [t_h^j, t_h^j + \delta_{\max}]$, where we have used (9). As a consequence, the first part of Condition (8) also holds, implying that $b_{ij}(s) = 1$ for every $s \in [t_h^j, t_h^j + \delta_{\max}]$. We get again

$$\int_{t_{ij}}^{\bar{t}_{ij}} a_{ji}(s)ds \geq \min\left(\frac{\mu}{n\Delta(0)}, 1\right) \int_{t_h^j}^{t_h^j + \delta_{\min}} b_{ij}(s)ds = \varepsilon,$$

which establishes that Assumption 3 holds in that case.

Suppose now that $a_{ij}(t) > 0$ because Condition (8) is satisfied at t for i, j . Then one can easily verify that Condition (7) was satisfied for j, i for all $s \in [t_k^j, t_k^j + \delta_{\min}]$ for some $t_k^j \in [t - \delta_{\max}, t]$, and an argument symmetric to that we have developed above shows that Assumption 3 also holds.

Since the weights $a_{ij}(t)$ are upper-bounded, applying Theorem 2 (or more precisely its direct extension to \mathbb{R}^2 , see Remark 1) shows that (i) the system converges: $x_i^* = \lim_{t \rightarrow \infty} x_i(t)$ exists for every i , and (ii) $x_i^* \neq x_j^*$ only if $\int_0^\infty a_{ij}(t)dt < \infty$.

To conclude the proof, suppose, to obtain a contradiction, that $\lim_{t \rightarrow \infty} \Delta(t) > d_1$, and thus that $\|x_i^* - x_j^*\| > d_1$ for some i, j . The continuity of x implies that $\|x_i(t) - x_j(t)\| > d_1$ for all $t > s$ for some s , and in particular for all $t_k^i > s$. It follows then from the *engage* rule (7) that $b_{ij}(t)$ would be set to 1 on infinitely many time intervals of length at least δ_{\min} . Besides, it follows from (11) that a_{ij} and b_{ij} remain within a bounded ratio, so that we would have $\int_0^\infty a_{ij}(t)dt = \infty$. However, we have seen that $x_i^* \neq x_j^*$ only if $\int_0^\infty a_{ij}(t)dt < \infty$, so there should hold $x_i^* = x_j^*$, in contradiction with our hypothesis. We have thus $\lim_{t \rightarrow \infty} \Delta(t) \leq d_1$. ■

Note that it is actually possible to have the robots converging to final positions within distances smaller than the d_1 from Proposition 4 from each other. This can be achieved by decreasing their maximal speed μ and the distance d_1 when approaching convergence. Such more evolved control laws are however out of the scope of this section, where our goal was to demonstrate the use of our results from Section II.

V. PROOFS

A. Proof of Theorem 1

Before we prove Theorem 1, we provide several intermediate results. Our proof uses the following result on cut-balance discrete-time consensus systems. This result is a special case of Theorem 1 in [22] restricted to deterministic systems.

Theorem 5: Let $y : \mathbb{N} \rightarrow \mathbb{R}^n$ be a solution to

$$y_i(p+1) = \sum_{j=1}^n b_{ij}(p)y_j(p), \quad (12)$$

where $b_{ij}(p) \geq 0$ and $\sum_{j=1}^n b_{ij}(p) = 1$. Suppose that the following assumptions hold:

a) *Lower bound on diagonal coefficients:* There exists a $\beta > 0$ such that $b_{ii}(p) \geq \beta$ for all i, p .

b) *Cut balance:* There exists a $K' > 0$ such that for every p and non-empty proper subset S of \mathcal{N} , there holds

$$\sum_{i \in S, j \notin S} b_{ij}(p) \leq K' \sum_{i \in S, j \notin S} b_{ji}(p). \quad (13)$$

Then, $y_i^* = \lim_{p \rightarrow \infty} y_i(p)$ exists for every i . Moreover, let $G' = (\mathcal{N}, E')$ be a directed graph where $(j, i) \in E'$ if $\sum_{p=0}^\infty b_{ij}(p) = +\infty$. There is a path from i to j in G' if and only if there is a path from j to i , and in that case there holds $y_i^* = y_j^*$.

Unlike certain results pre-dating those in [22] (e.g. Theorem 2 in [31]), Theorem 5 does not require the existence of a uniform lower bound on the positive coefficients b_{ij} , that is, the existence of a β' such that $b_{ij}(p) > 0 \Rightarrow b_{ij}(p) \geq \beta'$. This seemingly minor difference is actually essential for our purpose, as there is in general no such uniform lower bound in the context of our proof.

To apply Theorem 5, we focus on the values taken by the states at times t_p . Remember that the sequence of times t_p defines the intervals over which the integral reciprocity is satisfied.

Lemma 6: The sequence of states $(x(t_p))$ can be written as the trajectory of the discrete-time consensus system obtained by sampling (3)

$$x_i(t_{p+1}) = \sum_{j \in \mathcal{N}} \phi_{ij}(p) \cdot x_j(t_p), \quad (14)$$

where the weights $\phi_{ij}(p)$ are non-negative and satisfy $\sum_{j \in \mathcal{N}} \phi_{ij}(p) = 1$. This sampled system always exists and is unique for given weights $a_{ij}(t)$ and sampling times t_p . The weights $\phi_{ij}(p)$ are independent of states $x(t)$.

In particular, if $x_j(t_p) = 1$ for $j \in S$ and $x_k(t_p) = 0$ for $k \notin S$, for some $S \subseteq \mathcal{N}$, there holds

$$\sum_{j \in S} \phi_{ij}(p) = x_i(t_{p+1}). \quad (15)$$

Remark 3: The equality (15) provides a way of computing or bounding certain sums of the weights $\phi_{ij}(p)$ by considering the evolution of the systems starting from “artificial” states, where $x_j(t_p) = 1$ for some agents and $x_k(t_p) = 0$ for the others.

Note that these artificial states are only a formal tool to compute weights $\phi_{ij}(p)$, and their use does not result in any loss of generality.

Proof: Denote by $\Phi(t, T)$ the fundamental matrix of the linear dynamics (3) which is uniquely defined [32] by

$$x(T) = \Phi(t, T)x(t).$$

We define $\phi_{ij}(p)$ as the ij -th coefficient of matrix $\Phi(t_p, t_{p+1})$. So, the $\phi_{ij}(p)$ are unique and equation (14) is satisfied. Moreover, for given weights $a_{ij}(t)$, the matrix $\Phi(t, T)$ is independent of the state $x(t)$ and so are the weights $\phi_{ij}(p)$. So if we assume artificial states $x_j(t_p) = 1$ for $j \in S$ and $x_k(t_p) = 0$ for $k \notin S$, we obtain (15) from equation (14). And since system (3) preserves the nonnegativity of the states, it follows from equation (15) applied to $S = \{j\}$ that $\phi_{ij}(p) \geq 0$ for every i, j, p .

Finally, we can use the Peano-Baker formula [33] to show that $\sum_{j \in \mathcal{N}} \phi_{ij}(p) = 1$: the formula gives $\Phi(t, T)$ as the limit of a recursive series

$$\Phi(t, T) = \lim_{n \rightarrow \infty} M_n(T)$$

with

$$M_0(\tau) = I \text{ and } M_{n+1}(\tau) = I - \int_t^\tau L(s)M_n(s)ds,$$

where I is the identity matrix and $L(s)$ the Laplacian matrix of $A(s) = (a_{ij}(s))$, i.e. with diagonal elements equal to $\sum_{j \in \mathcal{N}} a_{ij}(s)$ and off-diagonal elements equal to $-a_{ij}(s)$. Since $L \cdot \mathbf{1} = 0$ with $\mathbf{1}$ the vector of all ones, we have from the recursive equation that $M_n \cdot \mathbf{1} = \mathbf{1}$ and by continuity, $\Phi(t, T) \cdot \mathbf{1} = \mathbf{1}$, thus $\sum_{j \in \mathcal{N}} \phi_{ij}(p) = 1$. ■

To obtain more insight on the discrete-time weights ϕ_{ij} , we give the next proposition which bounds the discrete-time weights ϕ_{ij} using the continuous-time weights a_{ij} . For concision, we will omit the explicit reference to time in a_{ij} when the context prevents any ambiguity.

Proposition 7: Under the uniform bound Assumption 2, we have for all proper subset of agents S and all $p \geq 0$,

$$G \cdot \sum_{\substack{i \in S \\ j \notin S}} \int_{t_p}^{t_{p+1}} a_{ij}(t)dt \leq \sum_{\substack{i \in S \\ j \notin S}} \phi_{ij}(p) \leq n \cdot \sum_{\substack{i \in S \\ j \notin S}} \int_{t_p}^{t_{p+1}} a_{ij}(t)dt,$$

with $G = \exp(-2nM)/n$.

Proof: Let $p \in \mathbb{N}$ and S a proper subset of \mathcal{N} . We assume that

$$\forall i \in S, x_i(t_p) = 0 \text{ and } \forall j \in S, x_j(t_p) = 1, \quad (16)$$

as suggested in Remark 3.

We first show the left inequality. We show that starting from state (16) at time t_p no agent $j \notin S$ can be arbitrarily close to 0 at time t_{p+1} . We have for all $\tau \in [t_p, t_{p+1}]$,

$$\begin{aligned} x_j(\tau) &= x_j(t_p) + \int_{t_p}^\tau \sum_{k \in \mathcal{N}} a_{jk}(t) \cdot (x_k(t) - x_j(t))dt \\ &\geq x_j(t_p) - \int_{t_p}^\tau \sum_{k \in \mathcal{N}} a_{jk}(t) \cdot x_j(t)dt, \end{aligned}$$

where we used $x_k(t) \geq 0, k \in \mathcal{N}$. We use Gronwall's inequality [34] and Assumption 2 (upper bound on interactions on each $[t_p, t_{p+1}]$) to obtain

$$j \notin S \Rightarrow x_j(\tau) \geq e^{-nM}, \forall \tau \in [t_p, t_{p+1}]. \quad (17)$$

We will use the bound (17) to establish that, due to attraction from agents not in S , the states $x_i(t_{p+1})$ of the agents $i \in S$ at time t_{p+1} are all at least a certain positive distance from 0.

Let now $h \in S$ be such that

$$\sum_{j \notin S} \int_{t_p}^{t_{p+1}} a_{hj}(t)dt = \max_{i \in S} \sum_{j \notin S} \int_{t_p}^{t_{p+1}} a_{ij}(t)dt,$$

i.e. agent h is the element in S receiving the highest influence from the rest of the group. There holds

$$\sum_{j \notin S} \int_{t_p}^{t_{p+1}} a_{hj}(t)dt \geq \frac{1}{n} \sum_{i \in S} \sum_{j \notin S} \int_{t_p}^{t_{p+1}} a_{ij}(t)dt. \quad (18)$$

Using the non-negativity $x_i \geq 0$ for $i \in S$ and the lower bound (17) on x_j for $j \notin S$, we have for all $\tau \in [t_p, t_{p+1}]$,

$$\begin{aligned} x_h(\tau) &= x_h(t_p) + \int_{t_p}^\tau \sum_{j \notin S} a_{hj}(x_j - x_h)dt + \int_{t_p}^\tau \sum_{k \in \mathcal{N}} a_{hk}(x_k - x_h)dt \\ &\geq x_h(t_p) + \int_{t_p}^\tau \sum_{j \notin S} a_{hj}x_jdt - \int_{t_p}^\tau \sum_{k \in \mathcal{N}} a_{hk}x_hdt. \\ &\geq e^{-nM} \int_{t_p}^\tau \sum_{j \notin S} a_{hj}dt - \int_{t_p}^\tau \sum_{k \in \mathcal{N}} a_{hk}x_hdt, \end{aligned}$$

where we have also used $x_h(t_p) = 0$. It follows then from Gronwall's inequality that

$$x_h(t_{p+1}) \geq e^{-nM} \int_{t_p}^{t_{p+1}} e^{-\int_\tau^{t_{p+1}} \sum_{k \in \mathcal{N}} a_{hk}ds} \sum_{j \notin S} a_{hj}d\tau. \quad (19)$$

The expression inside the exponential can be bounded using Assumption 2 (upper bound) together with (18). We have then

$$x_h(t_{p+1}) \geq \frac{1}{n} e^{-2nM} \sum_{i \in S, j \notin S} \int_{t_p}^{t_{p+1}} a_{ij}dt. \quad (20)$$

Moreover, $\phi_{ij}(p) \geq 0$ and equation (15) yield

$$\sum_{i \in S} \sum_{j \notin S} \phi_{ij}(p) \geq \sum_{j \notin S} \phi_{hj}(p) = x_h(t_{p+1}).$$

We conclude the first part of the proof combining the two previous equations.

We now turn to the second inequality. For $t \in [t_p, t_{p+1}]$, let $\bar{x}_S(t) = \max_{i \in S} x_i(t)$ be the largest value in x at time t , and $m(t)$ the index of (one of) the agents holding that largest value at time t . It was shown in [17, Proposition 2] that

$$\bar{x}_S(t) = \bar{x}_S(t_p) + \int_{t_p}^t \sum_{k \in \mathcal{N}} a_{m(\tau)k} (x_k - \bar{x}_S) d\tau.$$

Notice that the choice of state (16) implies $\bar{x}_S(t_p) = 0$. Since $x_j \leq 1$ for $j \notin S$ and $x_i \leq \bar{x}_S \leq 1$ for $i \in S$, we have

$$\begin{aligned} \bar{x}_S(t) &\leq \int_{t_p}^t \sum_{j \notin S} a_{m(\tau)j} (1 - \bar{x}_S) d\tau \\ &\leq \int_{t_p}^t \sum_{i \in S, j \notin S} a_{ij} (1 - \bar{x}_S) d\tau. \end{aligned}$$

Gronwall's inequality yields then

$$\begin{aligned} \bar{x}_S(t_{p+1}) &\leq 1 - \exp \left(- \int_{t_p}^{t_{p+1}} \sum_{i \in S, j \notin S} a_{ij} dt \right) \\ &\leq \int_{t_p}^{t_{p+1}} \sum_{i \in S, j \notin S} a_{ij} dt, \end{aligned} \quad (21)$$

from which we conclude

$$\sum_{i \in S, j \notin S} \phi_{ij}(p) = \sum_{i \in S} x_i(t_{p+1}) \leq n \bar{x}_S(t_{p+1}).$$

The previous proposition serves to transpose the cut-balance assumption provided in Theorem 1 to the discrete-time weights $\phi_{ij}(p)$. In particular, we can now show that the weights $\phi_{ij}(p)$ satisfy the condition of Theorem 5.

Lemma 8: Under the non-instantaneous reciprocity Assumption 1 and the uniform bound Assumption 2, the following properties hold :

- a) There exists a uniform lower bound $\beta > 0$ on diagonal elements: $\phi_{ii}(p) \geq \beta$, for all p and i .
- b) The weights $\phi_{ij}(p)$ satisfy the cut balance assumption (13) for some K' determined by the constants K and M of Assumptions 1 and 2.

Note that (b) would in general not be true for certain stronger forms of reciprocity. In particular, $\int_{t_p}^{t_{p+1}} a_{ij}(t) dt \leq K \int_{t_p}^{t_{p+1}} a_{ji}(t) dt$ does not imply the existence of a K' such that $\phi_{ij}(p) \leq K' \phi_{ji}(p)$.

Proof: The proof of (a) is as follows. For arbitrary $k \in \mathcal{N}$ and p , we suppose that $x_k(t_p) = 0$, and $x_i(t_p) = 1$ for every $i \neq k$. A reasoning similar to that leading to (17) in the proof of Proposition 7 shows that $x_k(t_{p+1}) \leq 1 - e^{-nM}$. It follows then from Lemma 6 applied to $S = \{k\}$ that $\sum_{j \in \mathcal{N}, j \neq k} \phi_{kj}(p) \leq 1 - e^{-nM}$, and thus that $\phi_{kk}(p) \geq e^{-nM}$, which establishes (a).

We now prove statement (b). The first inequality of Proposition 7 applied to S states that

$$\sum_{i \in S} \phi_{ij}(p) \leq n \cdot \sum_{i \notin S} \int_{t_p}^{t_{p+1}} a_{ij}(t) dt. \quad (22)$$

On the other hand, the second inequality of the same proposition applied to $\mathcal{N} \setminus S$ yields

$$G \cdot \sum_{i \notin S} \int_{t_p}^{t_{p+1}} a_{ij}(t) dt \leq \sum_{i \notin S} \phi_{ij}(p),$$

which can be rewritten as

$$G \cdot \sum_{i \in S} \int_{t_p}^{t_{p+1}} a_{ji}(t) dt \leq \sum_{i \in S} \phi_{ji}(p). \quad (23)$$

Statement (b) with $K' = n/G$ follows then directly from Assumption 1 and the inequalities (22) and (23). ■

of Theorem 1:

Since Lemma 8 is satisfied, Theorem 5 applies. Thus, the sequence $x(t_p)$ converges to some $x^* \in \mathbb{R}^n$. Denote by $G' = (\mathcal{N}, E')$ the directed graph where $(j, i) \in E$ if $\sum_{p=0}^{\infty} \phi_{ij}(p) = +\infty$. Theorem 5 implies that the (strongly) connected components of G' are entirely disconnected from each other (i.e. the different strongly connected components are not joined by any edge), and that $x_i^* = x_j^*$ if i and j belong to the same component. A local consensus takes thus place on each such component for the discrete-time system $y(p) = x(t_p)$. Now, the graph G of persistent interactions defined in the statement of Theorem 1 is in general different from G' . However, as a direct corollary of Proposition 7, we have that G and G' have the same connected components.

It remains to show that the continuous-time function $x(t)$ converges to the same x^* as the sequence $(x(t_p))$. We prove this by showing that for each set of node $S \subseteq \mathcal{N}$ inducing strongly connected component in G (or in G'), both the minimum $\underline{x}_S(t) = \min_{i \in S} x_i(t)$ and the maximum $\bar{x}_S(t) = \max_{i \in S} x_i(t)$ converge to the same value. Since S is a connected component of G , the integral influence $\sum_{i \in S, j \notin S} \int_0^{\infty} a_{ij}(t) dt$ is finite. For any $\mu > 0$, there exists some $T_\mu \geq 0$ such that

$$\sum_{i \in S, j \notin S} \int_{T_\mu}^{\infty} a_{ij}(t) dt < \mu.$$

We denote again by $m(\tau)$ the index of (one of) the agents with the largest value, so that $x_{m(\tau)}(\tau) = \bar{x}_S(\tau)$. Then, for

all $v > u \geq T_\mu$, we have

$$\begin{aligned}
\bar{x}_S(v) - \bar{x}_S(u) &\leq \sum_{j \notin S} \int_u^v a_{m(\tau)j} (x_j(\tau) - x_{m(\tau)}(\tau)) d\tau \\
&\leq \sum_{j \notin S} \int_u^v a_{m(\tau)j} |x_j(\tau) - x_{m(\tau)}(\tau)| d\tau \\
&\leq \sum_{i \in S, j \notin S} \int_u^v a_{m(\tau)j} |x_j(\tau) - x_i(\tau)| d\tau \\
&\leq \sum_{i \in S, j \notin S} \int_{T_\mu}^\infty a_{ij} |x_j(\tau) - x_i(\tau)| d\tau \\
&\leq \mu \Delta(0),
\end{aligned}$$

where $\Delta(0) = \max_{i \in \mathcal{N}} x_i(0) - \min_{i \in \mathcal{N}} x_i(0)$. This shows that the \bar{x}_S form a Cauchy sequence and thus that they converge. Since the sub-sequence $(\bar{x}_S(t_p))$ converges to x_i^* for some $i \in S$, there holds $\lim_{t \rightarrow +\infty} \bar{x}_S(t) = x_i^*$. We can apply the same reasoning to show that \bar{x}_S also converges $\lim_{p \rightarrow +\infty} \bar{x}_S(t_p) = x_i^*$. We conclude that for all $i \in S$, $x_i(t)$ converge to the same limit x_i^* . ■

B. Proof of Theorem 2

For concision, we say that an unordered pair $\{i, j\} = \{j, i\}$ is *active* over an interval I if $\int_{t \in I} a_{ij}(t) dt \geq \varepsilon$ and $\int_{t \in I} a_{ji}(t) dt \geq \varepsilon$. We let $T = \max_{i,j} T_{ij}$. The following observation compiles some properties that follow directly from the definition of being active.

Observation 1:

- Consider two intervals I, J with $I \subseteq J$. If $\{i, j\}$ is active over I , it is active over J .
- Under Assumption 3, if $a_{ij}(t) > 0$, then $\{i, j\}$ is active over $[t - T, t + T]$.
- Under Assumption 3, if $\{i, j\}$ is not active over $[t, t']$, then $a_{ij}(s) = a_{ji}(s) = 0$ for all $s \in [t + T, t' - T]$.

The next Proposition is the core of our proof, it allows building a sequence of times t_k valid for Assumptions 1 and 2.

Proposition 9: Suppose that Assumption 3 is satisfied, and let $M = M_1 + M_2$ where M_1, M_2 are any constant satisfying

$$M_2 > n(n-1)T + T \text{ and } M_1 \geq M_2 + T. \quad (24)$$

Then, there exists a sequence t_0, t_1, \dots with $t_0 = 0$, and $t_{k+1} - t_k \leq M$, such that the following condition A_k holds for every k .

A_k : for all $i, j \in \mathcal{N}$ distinct,
Condition $A1_k$ or Condition $A2_k$ holds,

with

$$\begin{cases} A1_k : \forall t \in [t_k, t_{k+1}], a_{ij}(t) = 0, \\ A2_k : \{i, j\} \text{ is active over } [t_k, t_{k+1}]. \end{cases}$$

The proof of Proposition 9 is based on an induction that makes use of the intermediate Condition B_k :

B_k : for all $i, j \in \mathcal{N}$ distinct,
Condition $B1_k$ or Condition $B2_k$ holds,

with

$$\begin{cases} B1_k : \forall t \in [t_k, t_k + T], a_{ij}(t) = 0, \\ B2_k : \{i, j\} \text{ is active over } [t_k, t_k + M_1]. \end{cases}$$

The next Lemma treats the initialization of the induction.

Lemma 10: Suppose that $a_{ij}(t) = 0$ for all $t \leq 0$, and let $t_0 = 0$. Then Condition B_0 holds.

Proof: Suppose that $B1_0$ does not hold, i.e. $a_{ij}(t) > 0$ for some $t \in [0, T]$. Then, Observation 1(b) implies that $\{i, j\}$ is active over $[t - T, t + T]$ which by Observation 1(a) implies that $\{i, j\}$ is *active* over $[\min(0, t - T), 2T]$. Since $a_{ij}(t') = 0$ for all $t' < 0$, it follows then that $\{i, j\}$ is active over $[0, 2T]$ and since $M_1 \geq M_2 + T > n(n-1) + 2T \geq 2T$ holds according to equation (24), $\{i, j\}$ is active over $[0, M_1]$ (again thanks to Observation 1(a)). Thus $B2_0$ holds and so does B_0 . ■

Proposition 11 (Inductive case): If there exists t_k such that Condition B_k holds, then there exists $t_{k+1} \leq t_k + M_1 + M_2$ for which Conditions A_k and B_{k+1} hold.

Proof:

Let us introduce the following two sets of unordered pairs of agents for every $t \in [t_k, t_k + M_1 + M_2]$.

- $R_t \subseteq \{\{i, j\} | i, j \in \mathcal{N}, i \neq j\}$: set of pairs $\{i, j\}$ which are active over time interval $[t_k, t]$.
- $V_t \subseteq \{\{i, j\} | i, j \in \mathcal{N}, i \neq j\}$: set of pairs $\{i, j\}$ for which $a_{ij}(t') = a_{ji}(t') = 0$ for all $t' \in [t, t_k + M_1 + M_2]$, i.e. set of pairs where there is no interaction between t and $t_k + M_1 + M_2$.

Note that for all $t_k \leq t \leq s \leq t_k + M_1 + M_2$, there holds $R_t \subseteq R_s$ and $V_t \subseteq V_s$, so that these sets are non-decreasing with time. The non-decrease of V_s is trivial while that of R_s follows directly from Observation 1(a).

For given k and t_k , we now build a t_{k+1} using Algorithm 1, which we prove to always successfully terminate. We first prove that Claims 1 and 2 hold, and then show how this implies the statement of Proposition 11.

Algorithm 1 Selection of t_{k+1}

Require: t_k satisfies B_k

Set $\bar{t} = t_k + M_1$

Switch over cases 0 to 3 :

Case 0: $\bar{t} \geq t_k + M_1 + M_2 - T$: STOP, FAILURE

Case 1: Conditions A_k and B_{k+1} are satisfied taking $t_{k+1} = \bar{t}$. STOP, SUCCESS.

Case 2: Condition A_k does not hold taking $t_{k+1} = \bar{t}$.

Claim 1: There exists $\{i, j\} \notin R_{\bar{t}}$ belonging to $R_{\bar{t}+T}$.

Then set $\bar{t} = \bar{t} + T$ and iterate.

Case 3: Condition B_{k+1} does not hold taking $t_{k+1} = \bar{t}$.

Claim 2: There exists $\{i, j\} \notin V_{\bar{t}}$ belonging to $V_{\bar{t}+T}$.

Then set $\bar{t} = \bar{t} + T$ and iterate.

Claim 1:

In Case 2 of Algorithm 1, Condition A_k does not hold. Thus, there exists $\{i, j\}$ such that $A1_k$ does not hold, i.e. $a_{ij}(t) > 0$ for some $t \in [t_k, \bar{t}]$, and $A2_k$ does not hold, i.e. $\{i, j\}$ is not active over $[t_k, \bar{t}]$.

The fact that $A2_k$ does not hold implies by definition of $R_{\bar{t}}$ that $\{i, j\} \notin R_{\bar{t}}$. Let us now show that $t \in [\bar{t} - T, \bar{t}]$. The fact that $A2_k$ does not hold together with Observation 1(c) implies that $a_{ij}(t') = 0$ for all $t' \in [t_k + T, \bar{t} - T]$. So either $t \in [t_k, t_k + T]$ or $t \in [\bar{t} - T, \bar{t}]$. We show that the first case is impossible: Since $\{i, j\}$ is not active over $[t_k, \bar{t}]$, and $\bar{t} \geq t_k + M_1$, the negation of Observation 1(a) implies that $\{i, j\}$ is not active over $[t_k, t_k + M_1]$, and thus that $B2_k$ does not hold. However, we know by hypothesis that B_k holds. Thus, $B1_k$ holds : $t \notin [t_k, t_k + T]$, and as a consequence, $t \in [\bar{t} - T, \bar{t}]$.

It follows then from Observation 1(b) that $\{i, j\}$ is active over $[t - T, t + T]$ and from Observation 1(a) that it is active over $[\bar{t} - 2T, \bar{t} + T]$. Since $\bar{t} \geq t_k + M_1 > t_k + 2T$, the pair $\{i, j\}$ is active over $[t_k, \bar{t} + T]$, so that : $\{i, j\} \in R_{\bar{t}+T}$, which achieves proving claim 1.

Claim 2:

Since Condition B_{k+1} does not hold, there is a pair $\{i, j\}$ that satisfies neither $B1_{k+1}$ nor $B2_{k+1}$, that is, one for which $a_{ij}(t) > 0$ for some $t \in [\bar{t}, \bar{t} + T]$, and for which $\{i, j\}$ is not active over $[\bar{t}, \bar{t} + M_1]$. Since $\bar{t} \leq t_k + M_1 + M_2 - T$ for otherwise we would have been in case 0, the $t \in [\bar{t}, \bar{t} + T]$ for which $a_{ij}(t) > 0$ lies in $[\bar{t}, t_k + M_1 + M_2]$, which implies that $\{i, j\} \notin V_{\bar{t}}$ by definition of $V_{\bar{t}}$. We now show that it belongs to $V_{\bar{t}+T}$.

By Observation 1(c), since $\{i, j\}$ is not active over $[\bar{t}, \bar{t} + M_1]$, there holds $a_{ij}(t') = a_{ji}(t') = 0$ for all $t' \in [\bar{t} + T, \bar{t} + M_1 - T]$. Also, by definition, $\bar{t} \geq t_k + M_1$ and $M_1 \geq M_2 + T$, so that

$$\bar{t} + M_1 - T \geq t_k + M_1 + M_1 - T \geq t_k + M_1 + M_2.$$

Thus, $a_{ij}(t') = a_{ji}(t') = 0$ for all $t' \in [\bar{t} + T, t_k + M_1 + M_2]$ and $\{i, j\} \in V_{\bar{t}+T}$.

To complete the proof of Proposition 11, we show that Algorithm 1 stops and that when it does, the choice $t_{k+1} = \bar{t}$ satisfies Conditions A_k and B_{k+1} . Indeed, at every iteration, either the algorithm stops, or case 2 or 3 applies and \bar{t} increases by T . In case 2, it follows from Claim 1 that the size of $R_{\bar{t}}$ increases by at least 1, and in case 3, it follows from Claim 2 that the size of $V_{\bar{t}}$ increases by at least 1. Since both $R_{\bar{t}}$ and $V_{\bar{t}}$ are sets of unordered pairs of distinct nodes, their size cannot exceed $n(n-1)/2$. Therefore, cases 2 and 3 do not apply more than $n(n-1)/2$ times each. In particular, case 0 or 1 must apply for some $\bar{t} \leq t_k + M_1 + n(n-1)T$ (remembering that \bar{t} is initially $t_k + M_1$), at which stage the algorithm stops. Now since according to equation (24), $M_2 > n(n-1)T + T$, case 0 or 1 apply for $\bar{t} < t_k + M_1 + M_2 - T$, so that case 1 must apply first, and the algorithm produces thus a t_{k+1} satisfying $t_{k+1} - t_k \leq M_1 + M_2$ for which A_k and B_{k+1} are satisfied. ■

The proof of Proposition 9 is then a direct consequence of Lemma 10 and Proposition 11.

of Theorem 2: We show that the sequence t_k built in Proposition 9 is valid for Assumptions 1 and 2. Observe first that since the $a_{ij}(t)$ are assumed to be uniformly bounded above and since $t_{k+1} - t_k \leq M$, there clearly holds $\int_{t_k}^{t_{k+1}} a_{ij}(t)dt < M'$ for some M' and all i, j and t_k , so that Assumption 2 holds. Moreover, it follows from Proposition 9 that either $a_{ij}(t) = a_{ji}(t) = 0$ for all $t \in [t_k, t_{k+1}]$, or $\int_{t_k}^{t_{k+1}} a_{ij}(t)dt \geq \varepsilon$ and $\int_{t_k}^{t_{k+1}} a_{ji}(t)dt \geq \varepsilon$. Since the latter integrals are also bounded by M' , there holds

$$\int_{t_k}^{t_{k+1}} a_{ij}(t)dt \leq \frac{M'}{\varepsilon} \int_{t_k}^{t_{k+1}} a_{ji}(t)dt,$$

which implies that Assumption 1 also holds. ■

VI. CONCLUSION

In this paper, we have developed a new reciprocity-based convergence result for continuous-time consensus systems. This result is based on a new assumption which allows for non-instantaneous reciprocity: Unlike previous studies, we only assume that reciprocity takes place on average over contiguous time intervals. This assumption is appropriate for various classes of systems (including classes of broadcasting, gossiping, and self-triggered system where communication is not necessarily synchronous). We have shown that integral reciprocity (Assumption 1) alone is not a sufficient condition for convergence. In particular, it does not forbid certain oscillatory behaviors. We have therefore proposed a companion assumption (Assumption 2) stating that quantity of interactions taking place in the intervals over which reciprocity occurs should be uniformly bounded. Under these two assumptions, we have proven that the trajectory of the system always converges, though not necessarily to consensus. Moreover, consensus takes place among agents in clusters of the graph of persistent interactions. We have also particularized our result to a class of systems satisfying a local pairwise form of reciprocity.

Apart from the integral reciprocity and uniform bound, our result does not make any assumption on the interactions between agents, and allows in particular for arbitrary long periods during which the system is idle. As a consequence, it is in general impossible to give absolute bounds on the speed of convergence under the assumptions that we have made. However, future works could relate the speed of convergence to the amount of interactions having taken place in the system, as in [19].

ACKNOWLEDGEMENT

The authors wish to thank Behrouz Touri for his help about Theorem 5.

REFERENCES

- [1] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

- [2] J. N. Tsitsiklis, "Problems in decentralized decision making and computation," *Ph.D. dissertation*, 1984.
- [3] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 169–182, Feb. 2005.
- [4] Z. Lin, M. Broucke, and B. Francis, "Local control strategies for groups of mobile autonomous agents," *IEEE Transactions on Automatic Control*, vol. 49, no. 4, pp. 622–629, 2004.
- [5] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Randomized gossip algorithms," *IEEE Transactions on Information Theory*, vol. 52, no. 6, pp. 2508–2530, 2006.
- [6] L. Xiao, S. Boyd, and S. Lall, "A scheme for robust distributed sensor fusion based on average consensus," in *Information Processing in Sensor Networks, 2005. IPSN 2005. Fourth International Symposium on*. IEEE, 2005, pp. 63–70.
- [7] J. C. Duchi, A. Agarwal, and M. J. Wainwright, "Dual averaging for distributed optimization: convergence analysis and network scaling," *IEEE Transactions on Automatic Control*, vol. 57, no. 3, pp. 592–606, 2012.
- [8] A. Nedic, A. Ozdaglar, and P. A. Parrilo, "Constrained consensus and optimization in multi-agent networks," *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 922–938, 2010.
- [9] B. Chazelle, "The convergence of bird flocking," *arXiv preprint arXiv:0905.4241*, 2009.
- [10] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," *Physical Review Letters*, vol. 75, no. 6, pp. 1226–1229, Aug 1995.
- [11] J. N. Lorenz, "Continuous opinion dynamics under bounded confidence: A survey," *International Journal of Modern Physics C*, vol. 18, no. 12, pp. 1819–1838, 2007.
- [12] C. Castellano, S. Fortunato, and V. Loreto, "Statistical physics of social dynamics," *Reviews of modern physics*, vol. 81, no. 2, p. 591, 2009.
- [13] F. Xiao and L. Wang, "Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays," *IEEE Transactions on Automatic Control*, vol. 53, no. 8, pp. 1804–1816, 2008.
- [14] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.
- [15] W. Ren, R. W. Beard, and E. M. Atkins, "A survey of consensus problems in multi-agent coordination," *American Control Conference*, pp. 1859–1864 vol. 3, 2005.
- [16] S. Manfredi and D. Angeli, "Frozen state conditions for asymptotic consensus of time-varying cooperative nonlinear networks," in *IEEE Conference on Decision and Control (CDC), 2013*, Dec 2013, pp. 1325–1330.
- [17] J. M. Hendrickx and J. N. Tsitsiklis, "Convergence of type-symmetric and cut-balanced consensus seeking systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 1, pp. 214–218, 2013.
- [18] C. De Persis, P. Frasca, and J. M. Hendrickx, "Self-triggered rendezvous of gossiping second-order agents," in *IEEE Conference on Decision and Control*, 2013.
- [19] S. Martin and A. Girard, "Continuous-time consensus under persistent connectivity and slow divergence of reciprocal interaction weights," *SIAM Journal on Control and Optimization*, vol. 51, no. 3, pp. 2568–2584, 2013.
- [20] J. M. Hendrickx and A. Olshevsky, "Symmetric continuum opinion dynamics: Convergence, but sometimes only in distribution," in *IEEE Conference on Decision and Control*, 2013.
- [21] G. Shi and K. H. Johansson, "The role of persistent graphs in the agreement seeking of social networks," *IEEE Journal on Selected Areas in Communications*, vol. 31, no. 9, pp. 595–606, September 2013.
- [22] B. Touri and C. Langbort, "On endogenous random consensus and averaging dynamics," *IEEE Transactions on Control of Network Systems*, vol. 1, no. 3, pp. 241–248, Sept 2014.
- [23] B. Touri and A. Nedić, "On backward product of stochastic matrices," *Automatica*, vol. 48, no. 8, pp. 1477–1488, 2012.
- [24] S. Bolouki and R. P. Malhame, "Ergodicity and class-ergodicity of balanced asymmetric stochastic chains," in *IEEE European Control Conference (ECC), 2013*, 2013, pp. 221–226.
- [25] S. Li and H. Wang, "Multi-agent coordination using nearest neighbor rules: Revisiting the vicsek model," *arXiv preprint cs/0407021*, 2004.
- [26] E. D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. New York: Springer, 1998.
- [27] S. Martin, A. Girard, A. Fazeli, and A. Jadbabaie, "Multiagent flocking under general communication rule," *IEEE Transactions on Control of Network Systems*, vol. 1, no. 2, pp. 155–166, June 2014.
- [28] L. Moreau, "Stability of continuous-time distributed consensus algorithms," in *IEEE Conference on Decision and Control (CDC)*, vol. 4, 2004, pp. 3998–4003.
- [29] A. Carullo and M. Parvis, "An ultrasonic sensor for distance measurement in automotive applications," *IEEE Sensors Journal*, vol. 1, no. 2, pp. 143–, Aug 2001.
- [30] F. Ceragioli, C. De Persis, and P. Frasca, "Discontinuities and hysteresis in quantized average consensus," *Automatica*, vol. 47, no. 9, pp. 1916–1928, 2011.
- [31] J. M. Hendrickx and J. N. Tsitsiklis, "A new condition for convergence in continuous-time consensus seeking systems," in *IEEE Conference on Decision and Control and European Control Conference (CDC-ECC), 2011*, 2011, pp. 5070–5075.
- [32] A. F. Filippov, *Differential equations with discontinuous righthand sides*. Dordrecht: Kluwer Academic Publishers Group, 1988, translated from the Russian.
- [33] R. W. Brockett, *Finite Dimensional Linear Systems. Decision and Control*. John Wiley & Sons, Inc., New York, 1970.
- [34] W. F. Ames and B. Pachpatte, *Inequalities for differential and integral equations*. Academic press, 1997, vol. 197.